

TOTAL ENERGY OF RADIAL MAPPINGS

SHAOLIN CHEN AND DAVID KALAJ

ABSTRACT. We prove that, the so called total energy functional defined on the class of radial stretchings between annuli attains its minimum on a total energy diffeomorphism between annuli on \mathbf{R}^n . This involves a subtle analysis of some special ODE. The result is an extension of the corresponding 2-dimensional case obtained by Iwaniec and Onninen (Arch. Ration. Mech. Anal., 194: 927-986, 2009).

1. INTRODUCTION

1.1. **Total energy.** Assume that $h \in \mathcal{W}^{1,n}$ is a homeomorphism between two annuli $\mathbb{A} = A(r, R)$ and $\mathbb{A}_* = A(r_*, R_*)$ of the Euclidean space \mathbf{R}^n . Then the total energy of h is defined by Iwaniec and Onninen in [8] by the formula

$$\mathcal{E}[h] = \frac{\alpha}{|\mathbb{A}_*|} \int_{\mathbb{A}} \|Dh\|^n + \frac{\beta}{|\mathbb{A}|} \int_{\mathbb{A}_*} \|Dh^{-1}\|^n,$$

$\alpha + \beta = 1$, $\alpha > 0$, $\beta > 0$. The functional

$$h \rightarrow \int_{\mathbb{A}} \|Dh\|^n$$

is called the n -energy functional while the functional

$$h \rightarrow \int_{\mathbb{A}_*} \|Dh^{-1}\|^n$$

is called the distortion functional. We define a radial stretching h as a mapping defined by a homeomorphism $H : [r, R] \mapsto [r_*, R_*]$ so that

$$h(x) = H(|x|) \frac{x}{|x|}.$$

In [8], Iwaniec and Onninen showed that the minimum of total energy for $n = 2$ attained by a stretching diffeomorphism of \mathbb{A} onto \mathbb{A}_* . One of their key steps was to solve the principal solution of so called equilibrium equation for the radial mappings. It is the following boundary value problem

$$(1.1) \quad \begin{cases} \ddot{H} = (H - t\dot{H}) \frac{(\alpha H \dot{H} + \beta t) \dot{H}^2}{(\alpha t \dot{H}^3 + \beta t) t H} \\ H(r) = r_*, \quad H(R) = R_*. \end{cases}$$

Namely they proved the following theorem.

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Theorem 1.1. ([8, Theorem 5.1]) *Given $R > r > 0$ and $R_* > r_* > 0$ there exists a unique strictly increasing function $H \in \mathcal{C}^\infty[r, R]$ that solves the equation (1.1) such that $H[r, R] = [r_*, R_*]$.*

Furthermore, in one of their main results ([8, Theorem 1.4]), they proved that the mapping $h(x) = H(|x|)\frac{x}{|x|}$ is the minimizer of the total energy functional (for $n = 2$). In [8, Theorem 1.5], they showed that this result cannot be extended to the Euclidean space \mathbb{R}^n for $n \geq 4$, however the case $n = 3$ remains an open problem.

In this paper, we extend Theorem 1.1 by proving the following theorem.

Theorem 1.2. *Let $n \geq 3$. If $\mathcal{P}(A, A_*)$ is the family of radial mappings with finite total energy, then there is a radial diffeomorphism $h = h_{\lambda_*}$ that minimizes the functional of total energy $\mathcal{E} : \mathcal{P}(A, A_*) \rightarrow \mathbf{R}$.*

The total energy is indeed a linear combination of the two operators, the energy functional and distortion functional. However it turns out that to minimize separately those two functionals do not solve the combination problem ([8]). The problem of finding a minimizer throughout certain class of homeomorphism has a long history. We want to refer here to some recent paper concerning minimization problem of harmonic Euclidean energy [5, 7] and of non-Euclidean energy [10] of heomorphisms between given annuli on Euclidean plane and on a Riemannian space respectively. Further, for minimization problem of distortion functional, we refer to the papers [1] and [11]. On the generalization of those problem for the spatial annuli and for n -harmonic energy (respectively (ρ, n) energy, see the papers [6] and [9].

2. THE PROOF OF MAIN RESULT

2.1. Hilbert norm of derivatives of the radial stretching and of its inverse. Assume that $h(x) = H(s)\frac{x}{s}$, where $s = |x|$. Let $\mathcal{H}(s) \stackrel{\text{def}}{=} \frac{H(s)}{s}$. Since $\text{grad}(s) = \frac{x}{|x|}$, we obtain

$$Dh(x) = (\mathcal{H}(s))' \frac{x \otimes x}{s} + \mathcal{H}(s)\mathbf{I},$$

where \mathbf{I} is the identity matrix. For $x \in \mathbb{A}$, let $T_1 = N = \frac{x}{|x|}$. Further, let T_2, \dots, T_n be $n - 1$ unit vectors mutually orthogonal and orthogonal to N . Thus

$$\begin{aligned} \|Dh(x)\|^2 &= \sum_{i=1}^n |Dh(x)T_i|^2 = \sum_{i=1}^n \left| (\mathcal{H}(s))' \frac{\langle T_i, x \rangle}{s} x + \mathcal{H}(s)T_i \right|^2 \\ &= (\mathcal{H}'(s))^2 s^2 + n(\mathcal{H}(s))^2 + 2\mathcal{H}(s)\mathcal{H}'(s)s \\ &= \frac{n-1}{s^2} H^2 + \dot{H}^2. \end{aligned}$$

Moreover, with respect to the basis T_i ($i = 1, \dots, n$), we have

$$D^*hDh = \begin{pmatrix} \dot{H}^2 & 0 & \dots & 0 \\ 0 & \frac{H^2}{s^2} & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & \frac{H^2}{s^2} \end{pmatrix}.$$

Here D^*h is the adjugate of the matrix Dh . Thus

$$J = J_h = \frac{\dot{H}H^{n-1}}{s^{n-1}}.$$

Let $X = Dh^{-1}(h(x))$. Then

$$XX^* = (D^*hDh)^{-1} = \begin{pmatrix} \dot{H}^{-2} & 0 & \dots & 0 \\ 0 & \frac{s^2}{H^2} & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & \frac{s^2}{H^2} \end{pmatrix},$$

which implies that

$$\|X\| = \sqrt{\dot{H}^{-2} + \frac{(n-1)s^2}{H^2}}.$$

Hence

$$\|Dh^{-1}(h(x))\|^n J_h = J_h \left[\dot{H}^{-2} + \frac{(n-1)s^2}{H^2} \right]^{\frac{n}{2}}.$$

2.2. Total energy of radial mappings. Let $a = \frac{\alpha}{|\mathbb{A}_*|}$ and $b = \frac{\beta}{|\mathbb{A}|}$. Then we calculate the energy of a radial stretching $h = H(r) \frac{x}{|x|}$. We obtain

$$\mathcal{E}[h] = a \int_{\mathbb{A}} \|Dh\|^n + b \int_{\mathbb{A}_*} \|Dh^{-1}\|^n = \omega_{n-1} \int_r^R \mathcal{L}[s, H, \dot{H}] ds,$$

where

$$\begin{aligned} \mathcal{L}[s, H, \dot{H}] &= as^{n-1} \|Dh(x)\|^n + b \|Dh^{-1}(h(x))\|^n J_h \\ &= as^{n-1} \left[\frac{(n-1)H^2}{s^2} + \dot{H}^2 \right]^{\frac{n}{2}} + bH^{n-1} \left[\frac{(n-1)s^2}{H^2} + \frac{1}{\dot{H}^2} \right]^{\frac{n}{2}} \dot{H} \end{aligned}$$

and ω_{n-1} is the Hausdoff measure of the unit sphere. Then the equilibrium equation (Euler-Lagrange equation) is

$$\mathcal{L}_H = \partial_s \mathcal{L}_{\dot{H}}$$

and it reduces to the equation

$$(2.1) \quad \ddot{H} = (s\dot{H} - H)M(s),$$

where

$$(2.2) \quad M(s) = \frac{\text{I} + \text{II}}{\text{III}},$$

$$\begin{aligned} \text{I} &= as^{-1} [(n-1)H^2 + s^2\dot{H}^2]^{\frac{n-4}{2}} [(n-1)H^2 + (n-2)sH\dot{H} + s^2\dot{H}^2], \\ \text{II} &= bH^{n-1} [(n-1)\dot{H}^2s^2 + H^2]^{\frac{n-4}{2}} \dot{H}^{1-n} [H^2 + (n-2)sH\dot{H} + (n-1)s^2\dot{H}^2] \\ \text{and} \end{aligned}$$

$$\begin{aligned} \text{III} &= (H^2 + s^2\dot{H}^2) \left\{ as[(n-1)H^2 + \dot{H}^2s^2]^{\frac{n-4}{2}} \right. \\ &\quad \left. + b \frac{H[(n-1)s^2\dot{H}^2 + H^2]^{\frac{n-4}{2}}}{\dot{H}^{1+n}} \right\}. \end{aligned}$$

So

$$(H - s\dot{H})' = -s(H - s\dot{H})M(s),$$

which is equivalent to

$$[\log(H - s\dot{H})]' = -sM(s).$$

Thus

$$\log(H - t\dot{H}) = \int_s^{s_1} (-\tau M(\tau)) d\tau + c,$$

which gives that

$$(2.3) \quad H - s\dot{H} = c \exp\left[\int_{s_1}^s (-\tau M(\tau)) d\tau\right].$$

Now we consider the following boundary problem

$$(2.4) \quad \begin{cases} \ddot{H} = (s\dot{H} - H)M(s) \\ H(r) = r_*, \quad H(R) = R_*. \end{cases}$$

which is the n -dimensional generalization of the boundary problem (1.1).

Now we prove that the diffeomorphic solution of (2.4) does exist. The idea is simple, we want to reduce the equation (2.4) into an ODE of the first order, but to do this we assume that the diffeomorphic solution H exists. This assumption is not harmful. Namely, the proof can be started from a certain first order ODE

$$(2.5) \quad F' = G[t, F(t)],$$

which has to do nothing with H (see (2.7) below). Then we solve (2.5) and, by using the solutions of it, we construct solutions of (2.4). Such a solution H will be a diffeomorphism and so it will satisfy one of the three cases listed below. On the other hand if we have a diffeomorphic solution H of (2.4), then it will satisfy the equation (2.3) for some continuous M and this will imply the uniqueness of solution H .

So if H is a strictly increasing C^2 diffeomorphism defined in a domain (a, b) that solves the equation (2.4), then $\dot{H}(s) > 0$ and from (2.3), we conclude that there are three possible cases:

- **Case 1** $c = 0$. Then $H - s\dot{H} \equiv 0$, or what is the same $H(s) = cs$, and this produces a linear mapping $h(x) = cx$, so in this case

$$\frac{R}{r} = \frac{R_*}{r_*}.$$

- **Case 2** $c > 0$. Then $H - s\dot{H} > 0$ and thus $t(s) = \frac{H(s)}{s}$ is a monotone increasing.
- **Case 3** $c < 0$. Then $H - s\dot{H} < 0$ and thus $t(s) = \frac{H(s)}{s}$ is a monotone decreasing.

If

$$\frac{R}{r} \neq \frac{R_*}{r_*},$$

we take the new variable $t = \frac{H(s)}{s}$ and the new function

$$(2.6) \quad F(t) = \dot{H} \left(\frac{H(s(t))}{s(t)} \right),$$

where $s(t)$ is the inverse of $t = t(s)$. Then we obtain

$$\ddot{H}(s) = \frac{\dot{F}(t)(s\dot{H} - H)}{s^2}$$

and

$$(2.7) \quad \dot{F}(t) = G[t, F(t)] = -\frac{U[t, F(t)] + V[t, F(t)]}{W[t, F(t)]},$$

where for $(t, y) \in \mathbf{R}_+^2$

$$U(t, y) \stackrel{\text{def}}{=} a \left[(n-1)t^2 + y^2 \right]^{\frac{n-4}{2}} \left[(n-1)t^2 + (n-2)ty + y^2 \right],$$

$$V(t, y) \stackrel{\text{def}}{=} b \frac{\left[(n-1)y^2 + t^2 \right]^{\frac{n-4}{2}} \left[t^2 + (n-2)ty + (n-1)y^2 \right]}{ty^{n-1}},$$

and

$$W(t, y) \stackrel{\text{def}}{=} (t^2 + y^2) \left\{ a \left[(n-1)t^2 + y^2 \right]^{\frac{n-4}{2}} + b \frac{t \left[(n-1)y^2 + t^2 \right]^{\frac{n-4}{2}}}{y^{1+n}} \right\}.$$

Observe that $G : \mathbf{R}_+^2 \rightarrow \mathbf{R}_-$ is smooth on \mathbf{R}_+^2 . Using the Picard-Lindelöf theorem, we observe that through any point $(t_0, y_0) \in \mathbf{R}_+^2$ there passes exactly one smooth integral curve defined in a neighborhood of this point. Then the extension theorem, see [3], tells us that such a local solution extends uniquely (as a solution) to the so-called maximal interval of existence. Denote this interval by (α, β) , where $0 \leq \alpha < \beta \leq \infty$.

The characteristic feature of the maximal interval of existence is that the points $(t, F(t))$ hit the boundary of \mathbf{R}_+^2 as $t \rightarrow \alpha$ or $t \rightarrow \beta$. Precisely, this means that the points $(t, F(t))$ lie outside any given compact subset of \mathbf{R}_+^2 when t approaches α or β .

Proposition 2.1. *Every local solution F of the equation (2.7) has $(0, +\infty)$ as the interval of its existence. Moreover, F is decreasing and*

- $\lim_{t \rightarrow 0} F(t) = +\infty$ and
- $\lim_{t \rightarrow +\infty} F(t) = 0$.

Proof. Since G is negative, we see that F is a decreasing function on its maximal interval (α, β) . Let $A = \lim_{t \downarrow \alpha} F(t)$ and $B = \lim_{t \uparrow \beta} F(t)$. We will show that $\alpha = 0$ and $\beta = \infty$. First, we prove $\beta = \infty$. Since $B < A$, we see that $B \in \mathbf{R}_+$ and thus the point (β, B) is a point of continuity of G which means that F can be continued smoothly above β . Hence $\beta = \infty$.

Now we prove $\alpha = 0$. If we assume $\alpha > 0$, then this assumption will lead to contradiction. Since F is decreasing, we know that $\lim_{t \downarrow \alpha} F(t) = +\infty$ and

$$\lim_{t \downarrow \alpha} F'(t) = \lim_{t \downarrow \alpha} G(t, F(t)) = -1.$$

But this is impossible, because the sequence

$$y_m = F(\alpha + 1) - F(\alpha + 1/m) = \int_{\alpha + \frac{1}{m}}^{\alpha + 1} F'(t) dt,$$

would be bounded.

Now we begin to show that $A = \infty$ and $B = 0$.

If $A = F(0) \stackrel{\text{def}}{=} \lim_{t \rightarrow 0} F(t) < \infty$, then, by (2.7), we obtain

$$(2.8) \quad \lim_{t \rightarrow 0} (tF'(t)) = -\frac{b}{a}(n-1)^{-1+\frac{n}{2}} A^{1-n}.$$

Then there is $\delta > 0$ so that $0 < t < \delta$ implies

$$tF'(t) \leq C = -\frac{b}{2a}(n-1)^{-1+\frac{n}{2}} A^{1-n}.$$

Thus

$$F(\delta) - F(t) = \int_t^\delta F'(t) \leq C \log\left(\frac{\delta}{t}\right),$$

and hence

$$F(t) \geq F(\delta) - C \log\left(\frac{\delta}{t}\right)$$

implying that $\lim_{t \rightarrow 0} F(t) = \infty$ which is a contradiction. Similarly, by using (2.7), we get

$$(2.9) \quad \lim_{t \rightarrow \infty} (tF'(t)) = -\frac{a}{b}(n-1)^{-1+\frac{n}{2}} B^{n+1},$$

where

$$B = \lim_{t \rightarrow +\infty} F(t)$$

and in similar way we establish the second statement. \square

Observe that the graph of the solution F intersects the diagonal $\{(x, x) : x > 0\}$ at exactly one point. Then we define the particular solution $F = F_\lambda$, with the initial condition $(\lambda, \lambda) \in \text{Graf}(F)$.

Now we prove the following

Proposition 2.2. *For fixed $t \in (0, \infty)$, the function $Q(\lambda) = F_\lambda(t)$ is an increasing C^1 function of $(0, \infty)$ onto itself.*

Proof. The fact that $Q(\lambda)$ is of class C^1 follows from the theorem on dependence of initial conditions and parameters ([3, Corollary 4.1.]).

Further, for two different $\lambda_1 < \lambda_2$, let $R(t) = F_{\lambda_2}(t) - F_{\lambda_1}(t)$. Then $R(t) \neq 0$, because near t , the Cauchy problem $F'(t) = G(t, F(t))$, $F(t) = F_0$ has the unique solution. Thus $R(t)$ has the constant sign. Further, because F_λ is decreasing, $F_{\lambda_2}(\lambda_2) = \lambda_2 > \lambda_1 = F_{\lambda_1}(\lambda_1) > F_{\lambda_1}(\lambda_2)$, it follows that $R(t) > 0$. Thus Q is increasing. In order to prove that $\lim_{\lambda \downarrow 0} Q(\lambda) = 0$ and $\lim_{\lambda \uparrow 0} Q(\lambda) = \infty$ do as follows. For $\epsilon < t < \epsilon^{-1}$, we obtain $\epsilon = F_\epsilon(\epsilon) \geq F_\epsilon(t)$ and $1/\epsilon = F_{1/\epsilon}(1/\epsilon) \leq F_{1/\epsilon}(t)$, because $t \rightarrow F_\lambda(t)$ is decreasing. This implies the proposition \square

Now we prove the following theorem which asserts that our boundary problem (2.4) has a unique diffeomorphic solution.

Theorem 2.3. *Let $0 < r < R$ and $0 < r_* < R_*$. Then there is an increasing diffeomorphism $H : [r, R] \rightarrow [r_*, R_*]$ that solves the principal equation (2.1).*

Proof. According to (2.6) and (2.7), the equation (2.1) is reduced to solving the initial value problems

$$(2.10) \quad \begin{cases} H'(s) = F_\lambda\left(\frac{H(s)}{s}\right), & \text{for } s > 0 \text{ and } H(s) > 0; \\ H(r) = r_*, \end{cases}$$

for $\lambda > 0$. Further we find $\lambda > 0$ so that $H(R) = R_*$. Using the Picard-Lindelöf theorem, we observe that for fixed λ_0 there is exactly one smooth solution $H = H_{\lambda_0}$ of the problem (2.10). Let (α, β) be the maximal interval of existence of H . Similarly as in the proof of Proposition 2.1, we obtain that $\beta = \infty$. Namely if $\beta < \infty$, then since F_{λ_0} is positive we conclude that $B = H(\beta^-) := \lim_{s \uparrow \beta} H(s) \in [r_*, \infty]$. If $B = \infty$, then $H'(\beta^-) = F_{\lambda_0}(+\infty) = 0$, and thus by (2.10), $H(\beta^-) \leq C\beta$ which is a contradiction. Thus $B < \infty$. But then $(x, y) = (\beta, B)$ is a point of continuity of $F_{\lambda_0}(\frac{y}{x})$, so the solution H can be extended above β . This implies that $\beta = \infty$. Since $r \in (\alpha, \beta)$, then $\alpha < r$. From (2.10) we obtain

$$H_\lambda(m) - r_* = \int_r^m F_\lambda\left(\frac{H_\lambda(s)}{s}\right) ds, \quad m > r$$

Assume that $H_{\lambda_n}(m) < M$ and $\lambda_n \rightarrow \infty$. Since F_λ is positive, we conclude that H_{λ_n} is increasing and

$$\frac{H_{\lambda_n}(s)}{s} \leq \frac{M}{r}, \quad r \leq s \leq m.$$

Thus

$$H_{\lambda_n}(m) - r_* \geq \int_r^m F_{\lambda_n} \left(\frac{M}{r} \right) ds.$$

The right hand-side of the last inequality tends to $+\infty$ because of Proposition 2.2. Thus $H_{\lambda}(m)$ tends to infinity when $\lambda \rightarrow \infty$.

Therefore, there is exactly one λ_* , such that $H = H_{\lambda_*}$ maps the interval $[r, R]$ onto $[r_*, R_*]$. \square

Next we prove Theorem 1.2, which is equivalent to the following theorem.

Theorem 2.4. *Let $n \geq 2$. If $\mathcal{P}(A, A_*)$ is the family of radial stretching with finite total energy, then the radial mapping $h = h_{\lambda_*}$, defined by $h(x) = H_{\lambda_*}(|x|) \frac{x}{|x|}$, minimizes the functional $\mathcal{E} : \mathcal{P}(A, A_*) \rightarrow \mathbf{R}$.*

Proof. Since we find the stationary point, which is unique, we only need to show that the given energy integral attains its minimum. First, we show that the function

$$\begin{aligned} \mathcal{L}[s, H, \dot{H}] &= as^{n-1} \left[\frac{(n-1)H^2}{s^2} + \dot{H}^2 \right]^{\frac{n}{2}} \\ &\quad + bH^{n-1} \left[\frac{(n-1)s^2}{H^2} + \frac{1}{\dot{H}^2} \right]^{\frac{n}{2}} \dot{H} \end{aligned}$$

is convex in \dot{H} .

For $K = \dot{H}$ we have the following

$$\begin{aligned} \partial_K \mathcal{L}[s, H, K] &= aKn \left[K^2 + \frac{H^2(n-1)}{s^2} \right]^{\frac{1}{2}(n-2)} s^{n-1} \\ &\quad - \frac{bH^{n-1}n \left[\frac{1}{K^2} + \frac{(n-1)s^2}{H^2} \right]^{\frac{1}{2}(n-2)}}{K^2} + bH^{n-1} \left[\frac{1}{K^2} + \frac{(n-1)s^2}{H^2} \right]^{\frac{n}{2}} \end{aligned}$$

and

$$\begin{aligned} \partial_{KK}^2 \mathcal{L}[s, H, K] &= (n-1)n (K^2 s^2 + H^2) \\ &\quad \times \left\{ a \frac{\left[H^2 + \frac{(n-1)s^2 K^2}{H^2} \right]^{\frac{n-4}{2}} H}{K^{1+n}} + bs \left[s^2 K^2 + (n-1)H^2 \right]^{\frac{n-4}{2}} \right\} \end{aligned}$$

which is clearly positive. Further we can find a positive constant C so that

$$(2.11) \quad C \left(|\dot{H}|^n + \frac{1}{|\dot{H}|^{n-1}} \right) \leq \mathcal{L}[s, H, \dot{H}],$$

which implies that the function L is coercive.

Let $h_m(x) = H_m[|x|]x/|x|$ be a sequence of smooth mappings with $H_m(r) = r_*$, $H_m(R) = R_*$ and

$$\inf_{h \in \mathcal{P}(A, A_*)} \mathcal{E}[h] = \lim_{m \rightarrow \infty} \mathcal{E}[h_m].$$

Then H_m are diffeomorphisms because $\dot{H}_m \neq 0$. Then up to a subsequence it converges to a monotone increasing function H_o . Moreover, since H_m is a bounded sequence of $\mathcal{W}^{1,n}$, it converges, up to a subsequence weakly to a mapping $H_o \in \mathcal{W}^{1,n}$.

By using the mentioned convexity of \mathcal{L} and the fact that \mathcal{L} is coercive, by standard theorem from the calculus of variation (as in the proof of [8, Theorem 1.2] (cf. [4, Theorem 63])), we obtain that

$$\mathcal{E}[h_o] = \lim_{m \rightarrow \infty} \mathcal{E}[h_m].$$

Further as $\mathcal{L}[s, H, K] \in C^\infty(\mathbf{R}_+^3)$, with $\partial_{KK}^2 \mathcal{L}[s, H, K] > 0$, we infer that $H_o \in C^\infty[r, R]$ (see [12, p. 17]) and H_o is the solution of our Euler-Lagrange equation. Thus it coincides with H_{λ^*} . \square

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S. L. CHEN, COLLEGE OF MATHEMATICS AND STATISTICS, HENGYANG NORMAL UNIVERSITY, HENGYANG, HUNAN 421008, PEOPLE'S REPUBLIC OF CHINA.

E-mail address: `mathechen@126.com`

D. KALAJ, UNIVERSITY OF MONTENEGRO, FACULTY OF NATURAL SCIENCES AND MATHEMATICS, CETINJSKI PUT B.B. 81000 PODGORICA, MONTENEGRO

E-mail address: `davidk@ac.me`